

Mixed Finite Element Methods
for Time Dependent Problems:
Application to Control

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MIXED FINITE ELEMENT METHODS FOR TIME DEPENDENT PROBLEMS: APPLICATION TO CONTROL.

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ABSTRACT

The main goal of this paper is to discuss mixed variational formulations for time dependent problems such as initial and boundary value problems for the heat and wave equations in a bounded domain Ω of $\mathbb{R}^N (N \geq 1)$. Then we shall use these formulations to derive mixed finite element approximations of the heat and wave equations. Finally, an application to an exact boundary controllability problem for the wave equation will be presented together with some numerical results. The techniques and application briefly considered here will be discussed with more details in a forthcoming paper.

INTRODUCTION

Mixed variational principles and the associated *finite element approximations* have proved to be very useful in order to derive accurate solution methods for boundary value problems for partial differential equations. This is particularly true for *elliptic problems* (see, e.g., [1], [2] and the references therein). A strong point of these techniques - compared to more traditional finite element methods - is that they give fairly *accurate approximations of the derivatives*; this last property is very interesting since in many problems one is more interested by the derivatives of a function than by the function itself. Mixed methods have also been applied to *time dependent problems* (see, e.g., [3], [17]) but there are indeed very few published papers and applications where these methods have been used for time dependent problems compared to the more classical finite element methods. Motivated by *optimal control* applications (cf. [4], [5]) we shall briefly discuss in this short article the following topics:

- (i) *Mixed variational formulations* for the heat and wave equations (Section 1.).
- (ii) *Mixed finite element approximations* of the heat and wave equations (Section 2.).
- (iii) An application to a *boundary control problem* for the wave equation (Section 3.).

1. MIXED VARIATIONAL FORMULATIONS FOR THE HEAT AND WAVE EQUATIONS.

1.1 Formulation of the basic time dependent problems.

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 1)$; we denote by Γ the boundary of Ω . Let T be a positive number (possibly equal to $+\infty$); we denote by Q and Σ the following sets of \mathbb{R}^{N+1} :

$$Q = \Omega \times (0, T), \Sigma = \Gamma \times (0, T).$$

We suppose now that physical phenomena are taking place on Ω , modelled by either the following *heat equation*

$$(1.1) \quad u_t - \Delta u = f \text{ in } Q,$$

$$(1.2) \quad u = g \text{ on } \Sigma,$$

$$(1.3) \quad u(x, 0) = u_o(x) \text{ on } \Omega,$$

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or by the following *wave equation*

$$(1.4) \quad u_{tt} - \Delta u = f \text{ in } Q,$$

$$(1.5) \quad u = g \text{ on } \Sigma,$$

$$(1.6) \quad u(x, 0) = u_o(x), \quad u_t(x, 0) = u_1(x) \text{ on } \Omega.$$

In (1.1) - (1.6) we have

$$x = \{x_i\}_{i=1}^N, u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, \Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}.$$

It follows from, e.g. [6], [7], that each of the two above problems has a unique solution provided that the data f and g belong to well chosen functional spaces. Since this paper is *engineering oriented* we shall not go into the details of those (Sobolev type) spaces for which the above problems are well-posed (there will be however some exceptions).

1.2 Mixed variational formulations for problems (1.1) – (1.3) and (1.4) – (1.6).

The key idea is to take $\nabla u (\nabla = \{\frac{\partial}{\partial x_i}\}_{i=1}^N)$ as *master variable* ; we introduce therefore a new unknown \mathbf{p} defined by

$$(1.7) \quad \mathbf{p} = \nabla u \text{ (in } Q \text{)}.$$

Assuming that u and \mathbf{p} are sufficiently smooth we obtain - *integrating by parts* with respect to the space variables - the following mixed variational formulations:

Mixed variational formulations of the heat equation (1.1) – (1.3) :

$$(1.8) \quad \int_{\Omega} (u_t - \nabla \cdot \mathbf{p} - f) v dx = 0, \quad \forall v \in L^2(\Omega), \text{ a.e. on } (0, T),$$

$$(1.9) \quad \int_{\Omega} (\mathbf{p} \cdot \mathbf{q} + u \nabla \cdot \mathbf{q}) dx = \int_{\Gamma} g \mathbf{q} \cdot \mathbf{n} d\Gamma, \quad \forall \mathbf{q} \in H(\Omega, \text{div}), \text{ a.e. on } (0, T),$$

$$(1.10) \quad u(x, 0) = u_o(x) \text{ on } \Omega.$$

Mixed variational formulations of the wave equation (1.4) – (1.6) :

$$(1.11) \quad \int_{\Omega} (u_{tt} - \nabla \cdot \mathbf{p} - f) v dx = 0, \quad \forall v \in L^2(\Omega), \text{ a.e. on } (0, T),$$

$$(1.12) \quad \int_{\Omega} (\mathbf{p} \cdot \mathbf{q} + u \nabla \cdot \mathbf{q}) dx = \int_{\Gamma} g \mathbf{q} \cdot \mathbf{n} d\Gamma, \quad \forall \mathbf{q} \in H(\Omega, \text{div}), \text{ a.e. on } (0, T),$$

$$(1.13) \quad u(x, 0) = u_o(x), \quad u_t(x, 0) = u_1(x).$$

In (1.8) - (1.13), we have used the following notation: $\mathbf{y} \cdot \mathbf{z} = \sum_{i=1}^N y_i z_i, \forall \mathbf{y}, \mathbf{z} \in \mathbb{R}^N$; \mathbf{n} is the unit vector of the outward normal at Γ ; $dx = dx_1 \cdots dx_N$ and finally

$$H(\Omega, \text{div}) = \{\mathbf{q} | \mathbf{q} \in L^2(\Omega), \nabla \cdot \mathbf{q} \in L^2(\Omega)\}.$$

2. MIXED FINITE ELEMENT APPROXIMATIONS OF THE HEAT AND WAVE EQUATIONS.

2.1 Generalities.

With h a *space discretization step*, we approximate $L^2(\Omega)$ and $H(\Omega, \text{div})$ by V_h and Q_h , respectively. We suppose that $V_h \subset L^2(\Omega)$, $Q_h \subset H(\Omega, \text{div})$ and also that V_h and Q_h satisfy compatibility conditions implying convergence properties for the corresponding approximations (see e.g., [1], [2] for details); an important condition to be satisfied is:

$$(2.1) \quad \nabla \cdot Q_h \subset V_h.$$

In the particular case where Ω is a 2 dimensional polygonal whose boundary is the union of segments parallel to the coordinate axis, we associate to Ω a "partition" R_h such that

$$(i) \quad R_h = \{K\}, \bar{\Omega} = \bigcup_{K \in R_h} \bar{K},$$

(ii) Each K is a rectangle whose edges are parallel to the coordinate axis,

(iii) If K and $K' \in R_h$, then $K \cap K' = \phi$, and either $\bar{K} \cap \bar{K}' = \phi$, or K and K' have only a whole edge or one vertex in common.

Following [1], [2] and [8] - [10], a *convergent* choice for V_h and Q_h , constructed from the above R_h , is given by:

$$(2.2) \quad V_h = \{v_h | v_h|_K \in Q_k, \forall K \in R_h\},$$

$$(2.3) \quad \begin{cases} Q_h = \{q_h | q_h = \{q_{ih}\}_{i=1}^2, q_h|_K \in (P_{k+1} \otimes P_k) \times (P_k \otimes P_{k+1}), \\ \forall K \in R_h; q_{ih} \text{ is continuous along the edges} \\ \text{of } R_h \text{ parallel to } 0x_{i+1}\}; \end{cases}$$

in (2.2), (2.3), k is a nonnegative integer, $Q_k = P_k \otimes P_k$, P_s is the space of the polynomials in one variable of degree $\leq s$, and $i+1$ has to be taken modulo 2. With such a choice for V_h and Q_h , condition (2.1) is clearly satisfied.

2.2 Discretization of the heat equation (1.1) - (1.3).

Semi - Discretization in space :

Using the spaces V_h and Q_h we shall "space discretize" (1.1) - (1.3), via (1.8) - (1.10) as follows:

Find a pair $\{u_h(t), p_h(t)\} \in V_h \times Q_h$, a.e. on $(0, T)$, such that

$$(2.4) \quad \int_{\Omega} \left(\frac{\partial u_h}{\partial t} - \nabla \cdot p_h - f_h \right) v_h dx = 0, \forall v_h \in V_h, \text{ a.e. on } (0, T),$$

$$(2.5) \quad \int_{\Omega} (p_h \cdot q_h + u_h \nabla \cdot q_h) dx = \int_{\Gamma} g_h q_h \cdot n d\Gamma, \forall q_h \in Q_h, \text{ a.e. on } (0, T),$$

$$(2.6) \quad u_h(0) = u_{oh}.$$

In (2.4) - (2.6), f_h, g_h and u_{oh} are convenient approximations of f, g and u_o , respectively (we can take, for example, u_{oh} as the L^2 -projection of u_o on V_h).

The above approximation is not practical since we still have to solve an ordinary differential system, or to be more precise a system, coupling ordinary differential equations and (linear) algebraic equations.

Full Discretization in space – time : Concentrating (for simplicity) on the *backward Euler* time discretization of (2.4) - (2.6) we finally obtain the following system of difference - algebraic equations (with $\Delta t(> 0)$ a *time discretization step*):

For $n \geq 0$, find $\{u_h^{n+1}, \mathbf{p}_h^{n+1}\} \in V_h \times Q_h$ such that

$$(2.7) \quad u_h^o = u_{oh},$$

$$(2.8) \quad \int_{\Omega} \left(\frac{u_h^{n+1} - u_h^n}{\Delta t} - \nabla \cdot \mathbf{p}_h^{n+1} - f_h^{n+1} \right) v_h dx = 0, \forall v_h \in V_h,$$

$$(2.9) \quad \int_{\Omega} (\mathbf{p}_h^{n+1} \cdot \mathbf{q}_h + u_h^{n+1} \nabla \cdot \mathbf{q}_h) dx = \int_{\Gamma} g_h^{n+1} \mathbf{q}_h \cdot \mathbf{n} d\Gamma, \forall \mathbf{q}_h \in Q_h.$$

From a practical point of view, we can easily eliminate u_h^{n+1} from (2.8), using the fact that $\nabla \cdot \mathbf{q}_h \in V_h$; we obtain then the following *linear variational equation* satisfied by \mathbf{p}_h^{n+1} :

$$(2.10) \quad \begin{cases} \int_{\Omega} (\Delta t \nabla \cdot \mathbf{p}_h^{n+1} \nabla \cdot \mathbf{q}_h + \mathbf{p}_h^{n+1} \cdot \mathbf{q}_h) dx = \int_{\Gamma} g_h^{n+1} \mathbf{q}_h \cdot \mathbf{n} d\Gamma \\ - \int_{\Omega} (u_h^n + \Delta t f_h^{n+1}) \nabla \cdot \mathbf{q}_h dx, \forall \mathbf{q}_h \in Q_h; \mathbf{p}_h^{n+1} \in Q_h. \end{cases}$$

Solving (2.10) can be done by a *direct method* - such as *Cholesky's* since the bilinear form in (2.10) is *symmetric* and *positive definite* - or by a *conjugate gradient algorithm* (see, for example, [11]). Once \mathbf{p}_h^{n+1} is known, computing u_h^{n+1} from (2.8) is straightforward.

Similarly, instead of backward Euler, we could have used schemes such as forward Euler, Crank - Nicholson, multisteps, Runge - Kutta,

2.3 Discretization of the wave equation (1.4) – (1.6).

Starting from the following variant of (2.4) - (2.6): Find a pair

$$\{u_h(t), \mathbf{p}_h(t)\} \in V_h \times Q_h, \text{ a.e. on } (0, T), \text{ such that}$$

$$(2.11) \quad \int_{\Omega} \left(\frac{\partial^2 u_h}{\partial t^2} - \nabla \cdot \mathbf{p}_h - f_h \right) v_h dx = 0, \forall v_h \in V_h, \text{ a.e. on } (0, T),$$

$$(2.12) \quad \int_{\Omega} (\mathbf{p}_h \cdot \mathbf{q}_h + u_h \nabla \cdot \mathbf{q}_h) dx = \int_{\Gamma} g_h \mathbf{q}_h \cdot \mathbf{n} d\Gamma, \forall \mathbf{q}_h \in Q_h, \text{ a.e. on } (0, T),$$

$$(2.13) \quad u_h(0) = u_{oh}, \frac{\partial u_h}{\partial t}(0) = u_{1h},$$

we can fully discretize the wave problem (1.4) - (1.6) by the following variant of the usual second order accurate, explicit finite difference discretization scheme of the wave equation:

Assuming that, for $n \geq 0$, u_h^n, \mathbf{p}_h^n and u_h^{n-1} are known compute first u_h^{n+1} as the solution of

$$(2.14) \quad \int_{\Omega} \left(\frac{u_h^{n+1} + u_h^{n-1} - 2u_h^n}{|\Delta t|^2} - \nabla \cdot \mathbf{p}_h^n - f_h^n \right) v_h dx = 0, \forall v_h \in V_h; u_h^{n+1} \in V_h,$$

and then \mathbf{p}_h^{n+1} as the solution of

$$(2.15) \quad \int_{\Omega} \mathbf{p}_h^{n+1} \cdot \mathbf{q}_h dx = \int_{\Gamma} g_h^{n+1} \mathbf{q}_h \cdot \mathbf{n} d\Gamma - \int_{\Omega} u_h^{n+1} \nabla \cdot \mathbf{q}_h dx, \forall \mathbf{q}_h \in Q_h; \mathbf{p}_h^{n+1} \in Q_h.$$

A most important step is clearly the *initialization* of scheme (2.14), (2.15); assuming that f, g, u_o, u_1 are sufficiently smooth we shall proceed as follows: compute $u_h^o, u_h^{-1}, \mathbf{p}_h^o$ and u_h^1

$$(2.16) \quad u_h^o = u_{oh}, \quad u_h^1 = u_h^{-1} + 2\Delta t u_{1h},$$

$$(2.17) \quad \begin{cases} \mathbf{p}_h^o \in Q_h, \\ \int_{\Omega} \mathbf{p}_h^o \cdot \mathbf{q}_h dx = \int_{\Gamma} g_h^o \mathbf{q}_h \cdot \mathbf{n} d\Gamma - \int_{\Omega} u_{oh} \nabla \cdot \mathbf{q}_h dx, \forall \mathbf{q}_h \in Q_h. \end{cases}$$

As shown in [12], $u_h(t)$ and $\mathbf{p}_h(t)$ will converge to $u(t)$ and $\nabla u(t)$ (u : solution of (1.4) - (1.6)) as h and $\Delta t \rightarrow 0$ if a *stability condition* such as

$$(2.18) \quad \Delta t \leq Ch$$

is satisfied.

Second order, unconditionally stable implicit variants of the above scheme can be obtained; they will be discussed in a following paper, together with applications to boundary control of the wave equation.

3. APPLICATION TO AN EXACT CONTROLLABILITY PROBLEM FOR THE WAVE EQUATION, VIA DIRICHLET BOUNDARY CONTROLS.

3.1 Formulation of the boundary control problem.

We follow here [4], [5]; we consider then a phenomenon taking place in Ω and modelled by the wave equation (we keep the notation of Section 1):

$$(3.1) \quad u_{tt} - \Delta u = 0 \text{ in } Q,$$

with the *initial conditions*

$$(3.2) \quad u(x, 0) = u_o(x), u_t(x, 0) = u_1(x) \text{ in } \Omega.$$

The problem here is to find g defined over $\Sigma (= \Gamma \times (0, T))$ such that the following *final conditions*

$$(3.3) \quad u(x, T) = 0, u_t(x, T) = 0 \text{ on } \Omega$$

hold if one has

$$(3.4) \quad u = g \text{ on } \Sigma$$

as *boundary condition*.

It has been proved by several authors (see [4], [5], [13] for references) that such a g exists provided that T is *sufficiently large* (the lower bound of the T 's for which (3.3) holds, $\forall u_o, u_1$, is - not surprisingly - of the order of diameter (Ω)).

3.2 Calculation of an exact Dirichlet control via the Hilbert Uniqueness Method of J. L. Lions

In [4], [5], J.L. Lions has introduced and analyzed a systematic way for constructing Dirichlet controls for which (3.3) holds. The construction technique is systematic and based on the *Hilbert Uniqueness Method* (HUM) to be briefly discussed below. From now on, we suppose that

$$(3.5) \quad u_o \in L^2(\Omega), u_1 \in H^{-1}(\Omega) (= (H_o^1(\Omega))'),$$

where

$$H_o^1(\Omega) = \{v | v \in L^2(\Omega), \frac{\partial v}{\partial x_i} \in L^2(\Omega), \forall i = 1, \dots, N, v = 0 \text{ on } \Gamma\},$$

$$H^{-1}(\Omega) \text{ is the dual space of } H_o^1(\Omega),$$

and we define E and E' by

$$(3.6) \quad E = H_o^1(\Omega) \times L^2(\Omega), E' = H^{-1}(\Omega) \times L^2(\Omega).$$

Next we define an operator $\Lambda \in L(E, E')$ as follows:

$$\begin{aligned} (i) & \quad \text{Take } \mathbf{e} = \{e_o, e_1\} \in E; \\ (ii) & \quad \text{Integrate from 0 to } T : \\ (3.7)_1 & \quad \phi_{tt} - \Delta \phi = 0 \text{ in } Q, \\ (3.7)_2 & \quad \phi = 0 \text{ on } \Sigma, \\ (3.7)_3 & \quad \phi(x, 0) = e_o(x), \phi_t(x, 0) = e_1(x) \text{ on } \Omega. \\ (iii) & \quad \text{Integrate from } T \text{ to } 0 : \\ (3.8)_1 & \quad \psi_{tt} - \Delta \psi = 0 \text{ in } Q, \\ (3.8)_2 & \quad \psi = \frac{\partial \phi}{\partial n} \text{ on } \Sigma, \\ (3.8)_3 & \quad \psi(x, T) = 0, \psi_t(x, T) = 0 \text{ on } \Omega. \\ (iv) & \quad \text{take} \\ (3.9) & \quad \Lambda \mathbf{e} = \{\psi_t(0), -\psi(0)\}, \end{aligned}$$

where $\psi(0)$ (resp. $\psi_t(0)$) is the function $x \rightarrow \psi(x, 0)$ (resp. $x \rightarrow \psi_t(x, 0)$).

It follows from J.L. Lions [4], [5] that $\Lambda \in L(E, E'), \forall T > 0$; moreover, if T is sufficiently large ($T > \text{diameter}(\Omega)$) then Λ is a *strongly elliptic* operator from E onto E' . In addition to these properties, Λ is *self-adjoint* and satisfies (with obvious notation):

$$(3.10) \quad \langle \Lambda \mathbf{e}, \mathbf{e}' \rangle = \int_{\Sigma} \frac{\partial \phi}{\partial n} \frac{\partial \phi'}{\partial n} d\Gamma dt, \forall \mathbf{e}, \mathbf{e}' \in E;$$

in (3.10), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E' and E which satisfies

$$\langle \Lambda \mathbf{e}, \mathbf{e}' \rangle = \int_{\Omega} (\Lambda \mathbf{e}) \cdot \mathbf{e}' dx$$

if Λe is sufficiently smooth.

Application to the exact boundary controllability of the wave equation :

(i) *Solve*

$$(3.11) \quad \Lambda e = \{u_1, -u_o\}.$$

(ii) *Solve (3.7), taking for e , in $(3.7)_3$, the solution of (3.11).*

(iii) *Take $g = \frac{\partial \phi}{\partial n}$ on Σ .*

If T is sufficiently large, it follows - from the properties of Λ - that (3.11) has a unique solution in E ; we have (cf. [4], [5]) $g \in L^2(\Sigma)$, and the corresponding solution of (3.8) satisfies (3.1) - (3.4), implying that g is a Dirichlet boundary control for which the exact controllability property (3.3) holds. Actually, of all the Dirichlet boundary control for which exact controllability holds, the one obtained by HUM, i.e. by solving (3.11) is the only one of *minimal norm* in $L^2(\Sigma)$, as shown in [4], [5]. From the properties of Λ , problem (3.11) can be solved by a *conjugate gradient algorithm* operating in space E ; such an algorithm is described in [13], [14], together with conforming finite element implementations of it.

3.3 Mixed formulation of the boundary control problem.

In fact, we shall describe a mixed formulation of problem (3.11):

Assuming that the initial data u_o and u_1 are sufficiently smooth, so that we can use integral representations, the problem is now to find a triple $\{e_o, p_o, e_1\}$ satisfying

$$(3.12) \quad \begin{cases} \{e_o, p_o\} \in W_o, e_1 \in L^2(\Omega); \forall \{v_o, \pi_o\} \in W_o, v_1 \in L^2(\Omega) \text{ we have} \\ \int_{\Omega} (\psi_t(0)v_o - \psi(0)v_1)dx = \int_{\Omega} (u_1v_o - u_ov_1)dx, \end{cases}$$

where in (3.12):

(i) The space W_o is defined by

$$(3.13) \quad \begin{cases} W_o = \{\{v_o, \pi_o\} | v_o \in L^2(\Omega), \pi_o \in (L^2(\Omega))^N, \int_{\Omega} (\pi_o \cdot q + v_o \nabla \cdot q)dx = 0, \\ \forall q \in H(\Omega, \text{div})\}; \end{cases}$$

it can be shown that

$$\{v_o, \pi_o\} \in W_o \leftrightarrow v_o \in H_o^1(\Omega), \pi_o = \nabla v_o.$$

(ii) $\psi(0)$ and $\psi_t(0)$ are obtained from e_o, p_o, e_1 as follows:

Integrate from 0 to T the *mixed formulated* following wave equation (cf. Section 2):

$$(3.14)_1 \quad \int_{\Omega} (\phi_{tt} - \nabla \cdot p)v dx = 0, \forall v \in L^2(\Omega), \text{ a.e. on } (0, T),$$

$$(3.14)_2 \quad \int_{\Omega} (p \cdot z + \phi \nabla \cdot z)dx = 0, \forall z \in H(\Omega, \text{div}), \text{ a.e. on } (0, T),$$

$$(3.14)_3 \quad \phi(x, 0) = e_o(x), \phi_t(x, 0) = e_1(x) \text{ on } \Omega;$$

then from T to 0 (using the fact that $\frac{\partial \phi}{\partial n} = \mathbf{p} \cdot \mathbf{n}$ on Σ):

$$(3.15)_1 \int_{\Omega} (\psi_{tt} - \nabla \cdot \mathbf{q}) v dx = 0, \forall v \in L^2(\Omega), \text{ a.e. on } (0, T),$$

$$(3.15)_2 \int_{\Omega} (\mathbf{q} \cdot \mathbf{z} + \psi \nabla \cdot \mathbf{z}) dx = \int_{\Gamma} \mathbf{p} \cdot \mathbf{n} \mathbf{z} \cdot \mathbf{n} d\Gamma, \forall \mathbf{z} \in H(\Omega, \text{div}), \text{ a.e. on } (0, T),$$

$$(3.15)_3 \psi(x, T) = 0, \psi_t(x, T) = 0 \text{ on } \Omega.$$

An easy calculation will show that (with obvious notation):

$$(3.16) \quad \begin{cases} \int_{\Omega} (\psi_t(0) e'_o - \psi(0) e'_1) dx = \int_{\Sigma} \mathbf{p} \cdot \mathbf{n} \mathbf{p}' \cdot \mathbf{n} d\Gamma dt, \\ \forall \{e_o, \pi_o; e_1\}, \{e'_o, \pi'_o; e'_1\} \in W_o \times L^2(\Omega). \end{cases}$$

From (3.16) it appears that the bilinear form occurring in (3.12) is *symmetric* and *positive semi definite*; actually, for T sufficiently large it is *strongly elliptic* (coercive) over $(W_o \times L^2(\Omega))^2$. From these properties, problem (3.12) can be solved by a *conjugate gradient algorithm* operating in $W_o \times L^2(\Omega)$; such an algorithm is described in Section 3.4.

3.4 Conjugate gradient solution of problem (3.12).

3.4.1. Generalities.

Problem (3.12) is a particular case of

$$(3.17) \quad \text{Find } u \in V \text{ such that } a(u, v) = L(v), \forall v \in V,$$

where in (3.17):

(i) V is an *Hilbert space*, equipped with the scalar product (\cdot, \cdot) , and the corresponding norm $\|\cdot\|$.

(ii) $a : V \times V \rightarrow \mathbb{R}$ is bilinear, continuous and V -elliptic (i.e. $\exists \alpha > 0$ such that $a(v, v) \geq \alpha \|v\|^2, \forall v \in V$).

(iii) $L : V \rightarrow \mathbb{R}$ is linear and continuous.

It is well known (cf., e.g., [15, Appendix 1]) that under the above hypotheses, problem (3.17) has a unique solution. If in addition to (i) - (iii), the bilinear form $a(\cdot, \cdot)$ is *symmetric* then problem (3.17) is equivalent to the following *minimization* one

$$(3.18) \quad \begin{cases} u \in V, \\ J(u) \leq J(v), \forall v \in V, \end{cases}$$

with $J(v) = \frac{1}{2} a(v, v) - L(v)$. Problem (3.17), (3.18) can then be solved by the following *conjugate gradient algorithm*:

Initialization

$$(3.19) \quad u^o \in V \text{ is given.}$$

Solve then

$$(3.20) \quad \begin{cases} g^o \in V, \\ (g^o, v) = a(u^o, v) - L(v), \forall v \in V. \end{cases}$$

If $g^o = 0$, or is "small", take $u = u^o$; if not, set

$$(3.21) \quad w^o = g^o.$$

Now for $n \geq 0$, suppose that u^n, g^n, w^n , are known with $w^n \neq 0$; define then $u^{n+1}, g^{n+1}, w^{n+1}$ as follows:

Descent : Compute

$$(3.22) \quad \rho_n = \|g^n\|^2 / a(w^n, w^n),$$

and

$$(3.23) \quad u^{n+1} = u^n - \rho_n w^n.$$

Test of the convergence and updating the descent direction : Solve

$$(3.24) \quad \begin{cases} g^{n+1} \in V, \\ (g^{n+1}, v) = (g^n, v) - \rho_n a(w^n, v), \forall v \in V. \end{cases}$$

If $g^{n+1} = 0$ - or is small - take $u = u^{n+1}$; if not compute

$$(3.25) \quad \gamma_n = \|g^{n+1}\|^2 / \|g^n\|^2,$$

and update w^n by

$$(3.26) \quad w^{n+1} = g^{n+1} + \gamma_n w^n.$$

Do $n = n + 1$ and go to (3.22).

The above algorithm converges, $\forall u^o \in V$, and we have (cf. [16]):

$$(3.27) \quad \|u^n - u\| \leq C \|u^o - u\| \left(\frac{\sqrt{\nu_a} - 1}{\sqrt{\nu_a} + 1} \right)^n,$$

where C is a constant, and where the condition number ν_a is given by

$$(3.28) \quad \nu_a = \sup_{v \in S} a(v, v) / \inf_{v \in S} a(v, v),$$

with $S = \{v | v \in V, \|v\| = 1\}$.

3.4.2 Application to the solution of problem (3.12)

Since problem (3.12) is a particular problem (3.17), with $V = W_o \times L^2(\Omega)$, it can be solved by the conjugate gradient algorithm (3.19) - (3.26). An important practical issue is the proper choice of the scalar product to be used over $W_o \times L^2(\Omega)$. A fairly convenient one is provided by

$$(3.29) \quad \begin{cases} \int_{\Omega} (v_o v'_o + \pi_o \cdot \pi'_o + v_1 v'_1) dx, \\ \forall \{v_o, \pi_o; v_1\}, \{v'_o, \pi'_o; v'_1\} \in W_o \times L^2(\Omega). \end{cases}$$

Applying algorithm (3.19) - (3.26) to the solution of problem (3.12), with $W_o \times L^2(\Omega)$ equipped with the scalar product (3.29), we obtain the following algorithm:

Initialization :

$$(3.30) \quad \{e_o^\circ, p_o^\circ\} \in W_o, e_1^\circ \in L^2(\Omega) \text{ are given.}$$

Integrate then from 0 to T the wave equation

$$(3.31)_1 \quad \int_{\Omega} (\phi_{tt}^\circ - \nabla \cdot p^\circ) v dx = 0, \forall v \in L^2(\Omega), \text{ a.e. on } (0, T),$$

$$(3.31)_2 \quad \int_{\Omega} (p^\circ \cdot z + \phi^\circ \nabla \cdot z) dx = 0, \forall z \in H(\Omega, \text{div}), \text{ a.e. on } (0, T),$$

$$(3.31)_3 \quad \phi^\circ(0) = e_o^\circ, \phi_t^\circ(0) = e_1^\circ.$$

Then from T to 0 :

$$(3.32)_1 \quad \int_{\Omega} (\psi_{tt}^\circ - \nabla \cdot q^\circ) v dx = 0, \forall v \in L^2(\Omega), \text{ a.e. on } (0, T),$$

$$(3.32)_2 \quad \int_{\Omega} (q^\circ \cdot z + \psi^\circ \nabla \cdot z) dx = \int_{\Gamma} p^\circ \cdot n \cdot z \cdot n d\Gamma, \forall z \in H(\Omega, \text{div}),$$

a.e. on $(0, T)$,

$$(3.32)_3 \quad \psi^\circ(T) = 0, \psi_t^\circ(T) = 0.$$

Compute then $\{g_o^\circ, \pi g_o^\circ\}$ and g_1° as follows: Solve the mixed elliptic problem:
Find $\{g_o^\circ, \pi g_o^\circ\} \in W_o$ such that

$$(3.33)_1 \quad \int_{\Omega} (g_o^\circ - \nabla \cdot \pi g_o^\circ) v dx = \int_{\Omega} (\psi_t^\circ(0) - u_1) v dx, \forall v \in L^2(\Omega),$$

$$(3.33)_2 \quad \int_{\Omega} (\pi g_o^\circ \cdot q + g_o^\circ \nabla \cdot q) dx = 0, \forall q \in H(\Omega, \text{div}),$$

and then

$$(3.34) \quad g_1^\circ = u_o - \psi^\circ(0).$$

If $\{g_o^\circ, \pi g_o^\circ\} = \{0, 0\}$, $g_1^\circ = 0$, or are small, take $p^\circ \cdot n|_{\Sigma}$ as boundary control; if not, set

$$(3.35) \quad \{w_o^\circ, \pi w_o^\circ; w_1^\circ\} = \{g_o^\circ, \pi g_o^\circ; g_1^\circ\}.$$

Then for $n \geq 0$, assuming that $\{e_o^n, p_o^n\}, e_1^n, \phi^n, \psi^n, \{g_o^n, \pi g_o^n\}, g_1^n, \{w_o^n, \pi w_o^n\}, w_1^n$ are known, we compute $\{e_o^{n+1}, p_o^{n+1}\}, e_1^{n+1}, \phi^{n+1}, \psi^{n+1}, \{g_o^{n+1}, \pi g_o^{n+1}\}, g_1^{n+1}, \{w_o^{n+1}, \pi w_o^{n+1}\}, w_1^{n+1}$, as follows :

Descent :

Integrate from 0 to T

$$(3.36)_1 \quad \int_{\Omega} (\bar{\phi}_{tt}^n - \nabla \cdot \bar{\mathbf{p}}^n) v dx = 0, \forall v \in L^2(\Omega), \text{ a.e. on } (0, T),$$

$$(3.36)_2 \quad \int_{\Omega} (\bar{\mathbf{p}}^n \cdot \mathbf{z} + \bar{\phi}^n \nabla \cdot \mathbf{z}) dx = 0, \forall \mathbf{z} \in H(\Omega, \text{div}), \text{ a.e. on } (0, T),$$

$$(3.36)_3 \quad \bar{\phi}^n(0) = w_o^n, \bar{\phi}_t^n(0) = w_1^n.$$

Then from T to 0 :

$$(3.37)_1 \quad \int_{\Omega} (\bar{\psi}_{tt}^n - \nabla \cdot \bar{\mathbf{q}}^n) v dx = 0, \forall v \in L^2(\Omega),$$

$$\text{a.e. on } (0, T),$$

$$(3.37)_2 \quad \int_{\Omega} (\bar{\mathbf{q}}^n \cdot \mathbf{z} + \bar{\psi}^n \nabla \cdot \mathbf{z}) dx = \int_{\Gamma} \bar{\mathbf{p}}^n \cdot \mathbf{n} \mathbf{z} \cdot \mathbf{n} d\Gamma, \forall \mathbf{z} \in H(\Omega, \text{div}),$$

$$\text{a.e. on } (0, T),$$

$$(3.37)_3 \quad \bar{\psi}^n(T) = 0, \bar{\psi}_t^n(T) = 0$$

Solve now the mixed elliptic problem: Find $\{\bar{g}_o^n, \pi \bar{g}_o^n\} \in W_o$ such that

$$(3.38)_1 \quad \int_{\Omega} (\bar{g}_o^n - \nabla \cdot \pi \bar{g}_o^n) v dx = \int_{\Omega} \bar{\psi}_t^n(0) v dx, \forall v \in L^2(\Omega),$$

$$(3.38)_2 \quad \int_{\Omega} (\pi \bar{g}_o^n \cdot \mathbf{q} + \bar{g}_o^n \nabla \cdot \mathbf{q}) dx = 0, \forall \mathbf{q} \in H(\Omega, \text{div}),$$

and set

$$(3.39) \quad \bar{g}_1^n = -\bar{\psi}^n(0).$$

Compute now

$$(3.40) \quad \left\{ \begin{aligned} \rho_n &= \frac{\int_{\Omega} (|g_o^n|^2 + |\pi g_o^n|^2 + |g_1^n|^2) dx}{\int_{\Omega} (w_o^n \bar{\psi}_t^n(0) - w_1^n \bar{\psi}^n(0)) dx} \\ &= \frac{\int_{\Omega} (|g_o^n|^2 + |\pi g_o^n|^2 + |g_1^n|^2) dx}{\int_{\Omega} (\bar{g}_o^n w_o^n + \pi \bar{g}_o^n \cdot \pi w_o^n + \bar{g}_1^n w_1^n) dx} \end{aligned} \right.$$

and then

$$(3.41) \quad \{e_o^{n+1}, \mathbf{p}_o^{n+1}, e_1^{n+1}\} = \{e_o^n, \mathbf{p}_o^n, e_1^n\} - \rho_n \{w_o^n, \pi w_o^n, w_1^n\},$$

$$(3.42) \quad \{\phi^{n+1}, \mathbf{p}^{n+1}\} = \{\phi^n, \mathbf{p}^n\} - \rho_n \{\bar{\phi}^n, \bar{\mathbf{p}}^n\},$$

$$(3.43) \quad \{\psi^{n+1}, \mathbf{q}^{n+1}\} = \{\psi^n, \mathbf{q}^n\} - \rho_n \{\bar{\psi}^n, \bar{\mathbf{q}}^n\},$$

$$(3.44) \quad \{g_o^{n+1}, \pi g_o^{n+1}, g_1^{n+1}\} = \{g_o^n, \pi g_o^n, g_1^n\} - \rho_n \{\bar{g}_o^n, \pi \bar{g}_o^n, \bar{g}_1^n\}.$$

Test of the convergence. New descent Direction :

If $\{g_o^{n+1}, \pi g_o^{n+1}, g_1^{n+1}\} = \{0, 0, 0\}$ - or is small - take $\mathbf{p}^{n+1} \cdot \mathbf{n}|_{\sum}$ as boundary control; if not compute

$$(3.45) \quad \gamma_n = \frac{\int_{\Omega} (|g_o^{n+1}|^2 + |\pi g_o^{n+1}|^2 + |g_1^{n+1}|^2) dx}{\int_{\Omega} (|g_o^n|^2 + |\pi g_o^n|^2 + |g_1^n|^2) dx}$$

and then

$$(3.46) \quad \{w_o^{n+1}, \pi w_o^{n+1}, w_1^{n+1}\} = \{g_o^{n+1}, \pi g_o^{n+1}, g_1^{n+1}\} + \gamma_n \{w_o^n, \pi w_o^n, w_1^n\}.$$

Do $n = n + 1$ and go to (3.36).

Remark 3.1 : Problems (3.33) and (3.38) are particular cases of

$$(3.47)_1 \quad \int_{\Omega} (u - \nabla \cdot \mathbf{p}) v dx = \int_{\Omega} f v dx, \forall v \in L^2(\Omega),$$

$$(3.47)_2 \quad \int_{\Omega} (\mathbf{p} \cdot \mathbf{q} + u \nabla \cdot \mathbf{q}) dx = 0, \forall \mathbf{q} \in H(\Omega, \text{div}),$$

which is the mixed formulation of the following *Dirichlet problem*

$$(3.48) \quad -\Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma.$$

Observing that $\nabla \cdot \mathbf{q} \in L^2(\Omega), \forall \mathbf{q} \in H(\Omega, \text{div})$, we can eliminate u from (3.47)₁, (3.47)₂ to obtain that \mathbf{p} satisfies (if $f \in L^2(\Omega)$) :

$$(3.49) \quad \begin{cases} \mathbf{p} \in H(\Omega, \text{div}), \\ \int_{\Omega} (\nabla \cdot \mathbf{p} \nabla \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{q}) dx = - \int_{\Omega} f \nabla \cdot \mathbf{q} dx, \forall \mathbf{q} \in H(\Omega, \text{div}). \end{cases}$$

Solving (3.49) (in fact its discrete variants) is fairly easy and can be done by conjugate gradient algorithms (see, e.g., [9] for details). Once \mathbf{p} is known, one obtains easily u from (3.47)₁. Combining the above algorithm with the mixed finite element approximations and time discretization schemes of the wave equation discussed in Section 2 is (almost) straightforward; this issue will be discussed in a forthcoming paper.

3.4.3. Numerical experiments.

The mixed finite element approximation and time discretization schemes of the wave equation, described in Section 2, have been combined to algorithm (3.30) - (3.46), to solve problem (3.11) when $\Omega = (0, 1) \times (0, 1)$ and $T = 2\sqrt{2}$. Using the Fourier series techniques described in [13] we have computed those initial data u_o and u_1 for which the solution $\mathbf{e} (= \{e_o, e_1\})$ of (3.11) is given by

$$(3.50) \quad e_o(x_1, x_2) = \sin \pi x_1 \sin \pi x_2, \quad e_1 = \pi \sqrt{2} e_o.$$

We have used the mixed finite element approximations of Section 2, with $k = 1$ and R_h the regular partition of Ω associated to the vertices $\{ih, jh\}$ with $0 \leq i, j \leq N$, N being an integer such that $Nh = 1$; we have taken $N = 16, 32, 64$. The

time discretization of the various wave equations involved in the calculations was obtained using the (conditionally stable) explicit scheme described in Section 2. Obtaining the (approximate) values of the control $\frac{\partial \phi}{\partial n} = \mathbf{p} \cdot \mathbf{n}$ on Σ , was quite easy since the values of the fluxes (i.e. of the normal components of \mathbf{p}_h), at the element interfaces and at the boundary Γ , are the natural *degrees of freedom* for the functions belonging to the finite dimensional space Q_h approximating $H(\Omega, \text{div})$.

For $h = 1/16$ (resp. $1/32, 1/64$) the finite dimensional variant of algorithm (3.30) - (3.46) converges in 48 (resp. 72, 119) iterations (the number of iterations varies - approximately - like \sqrt{N}). These numbers are much higher than those obtained in [13], where the space approximation was achieved by a conforming finite element method, coupled to a biharmonic Tychonoff regularization to eliminate spurious oscillations. On the other hand, using, as in the present paper, mixed finite element approximations, it is not necessary to use regularization to obtain very good numerical results, as shown in Figures 3.1 (a), (b), (c) ($N=6$), 3.2 (a), (b), (c) ($N=32$), 3.3(a), (b), (c) ($N=64$).

Figures (a) (resp. (b)) show the variation of the exact (-) and computed $(\dots)e_0$ (resp. e_1), for $0 \leq x_1 \leq 1, x_2 = .5$. Figures (c) show the variation on $(0, T)$ of the $L^2(\Gamma)$ - norm of the exact and approximate boundary controls.

All the above calculations have been done on a CRAY X-MP supercomputer.

4. CONCLUSION.

In this paper we have discussed the application of *mixed finite element methods* to the numerical solution of direct or inverse problems for time dependent equations. These mixed methods are robust and accurate. They are however more complicated to implement than the traditional finite element methods. Indeed many important issues remain concerning the practical use of the mixed methods considered here, such as speeding up calculations by multigrid and/or domain decomposition methods (cf. [10]); we intend to investigate them in the near future.

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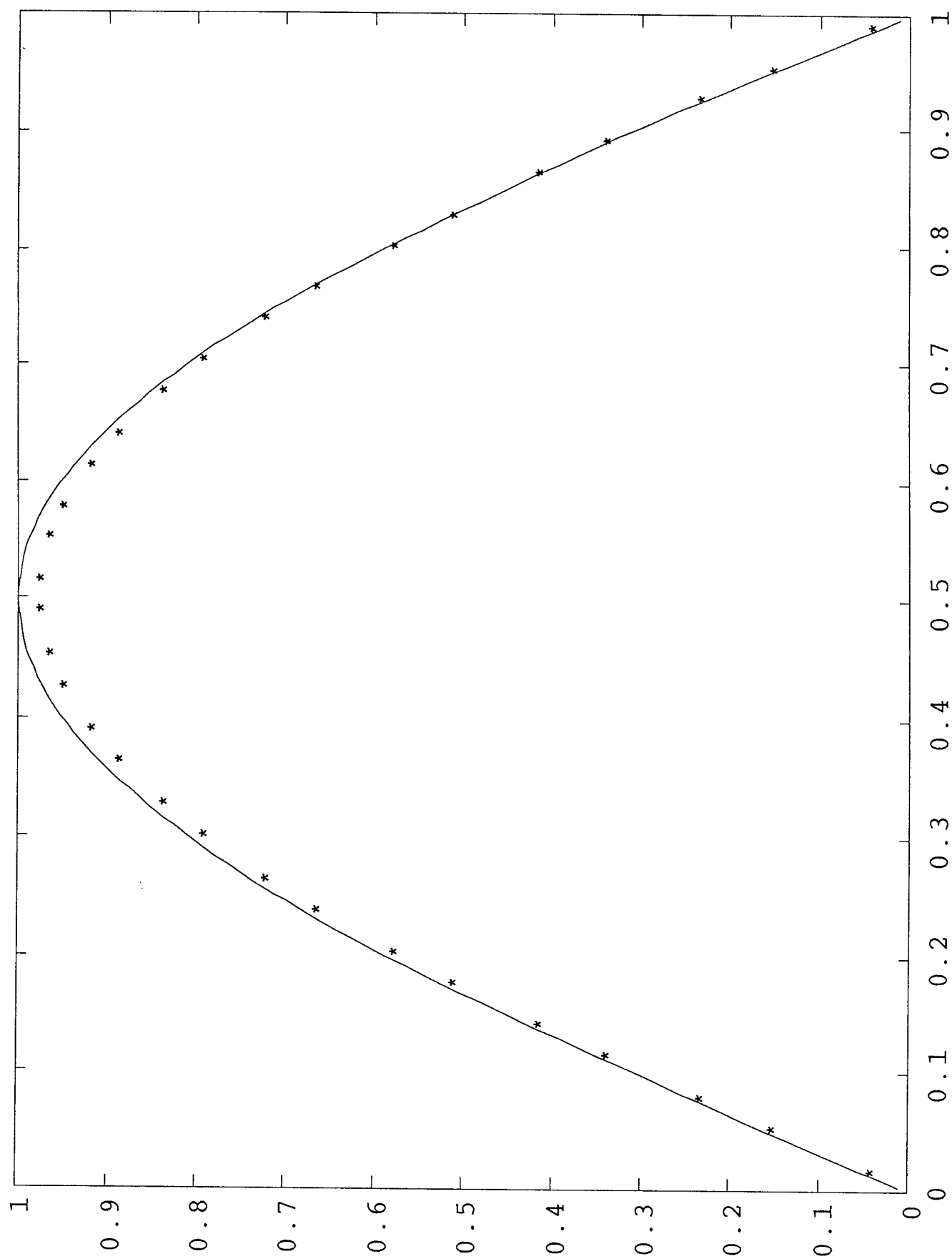


Figure 3.1 (a): Comparison between exact (-) and computed (*) value of $e_0(x_1, .5)$ ($h=1/16$).

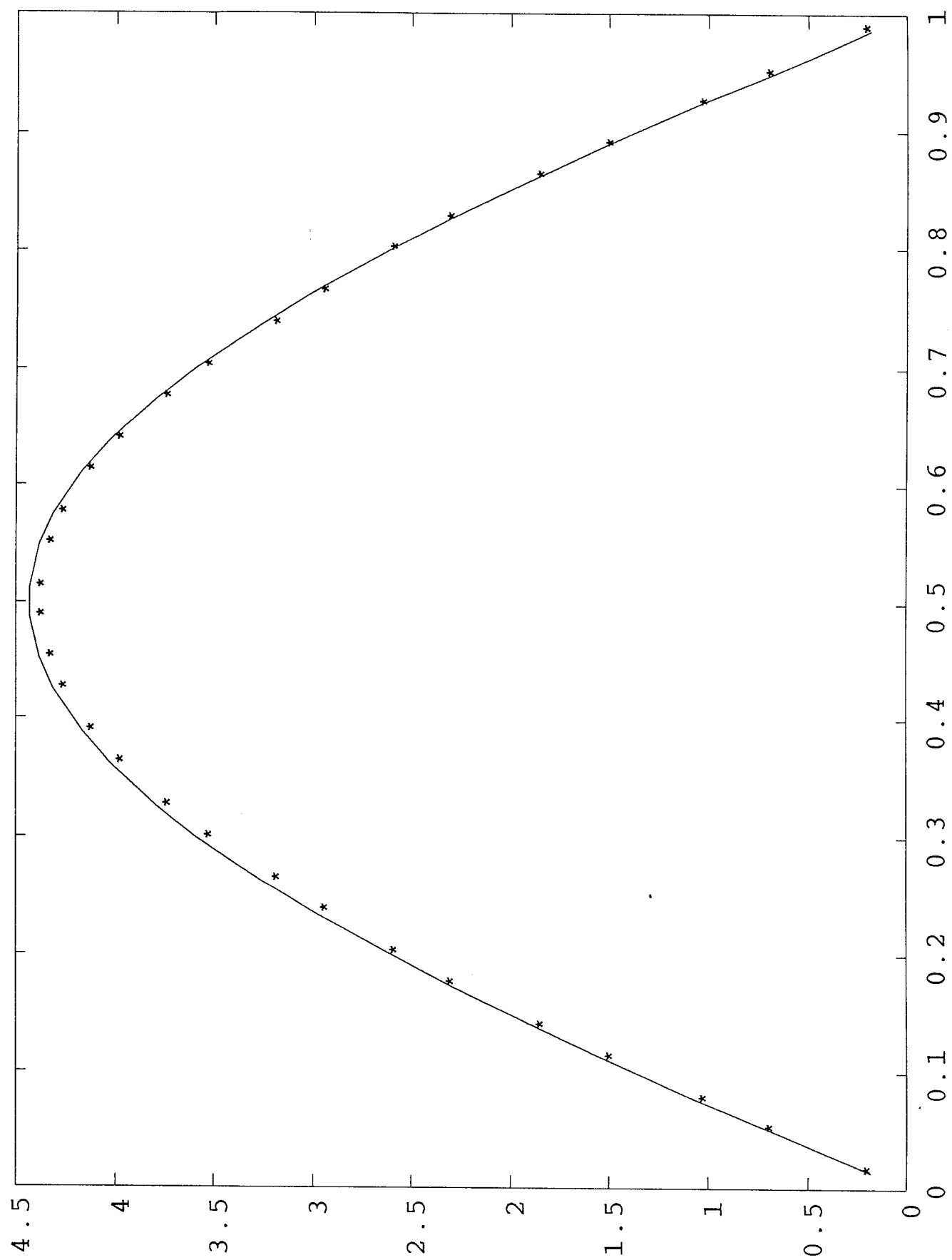
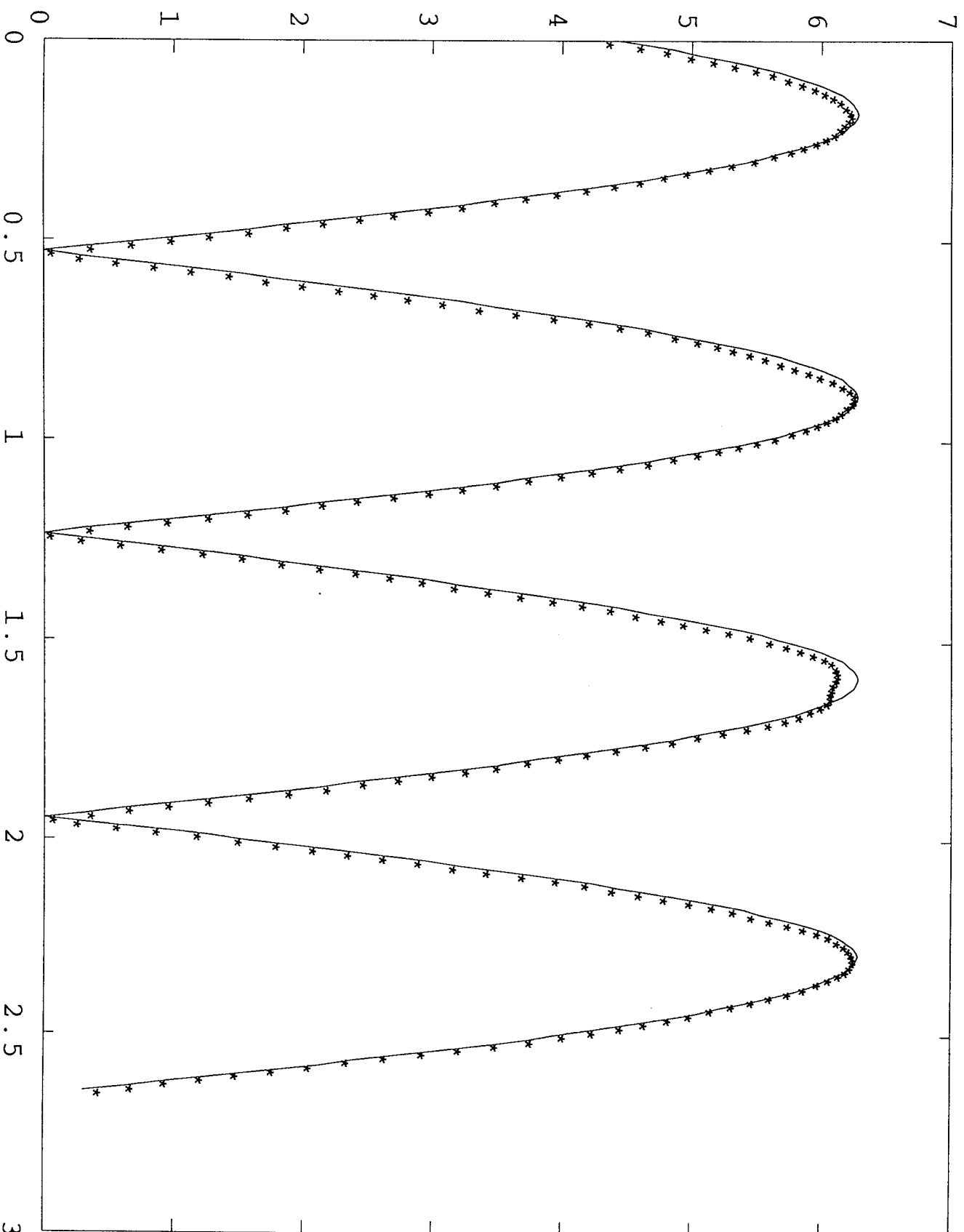


Figure 3.1 (b): Comparison between exact (-) and computed (*) value of $e_1(x_1, .5)$ ($h=1/16$).



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Figure 3.1 (c): Comparison between exact (-) and computed (*) value of $\|g(t)\|_{L_2(G)}$ ($h=1/16$).

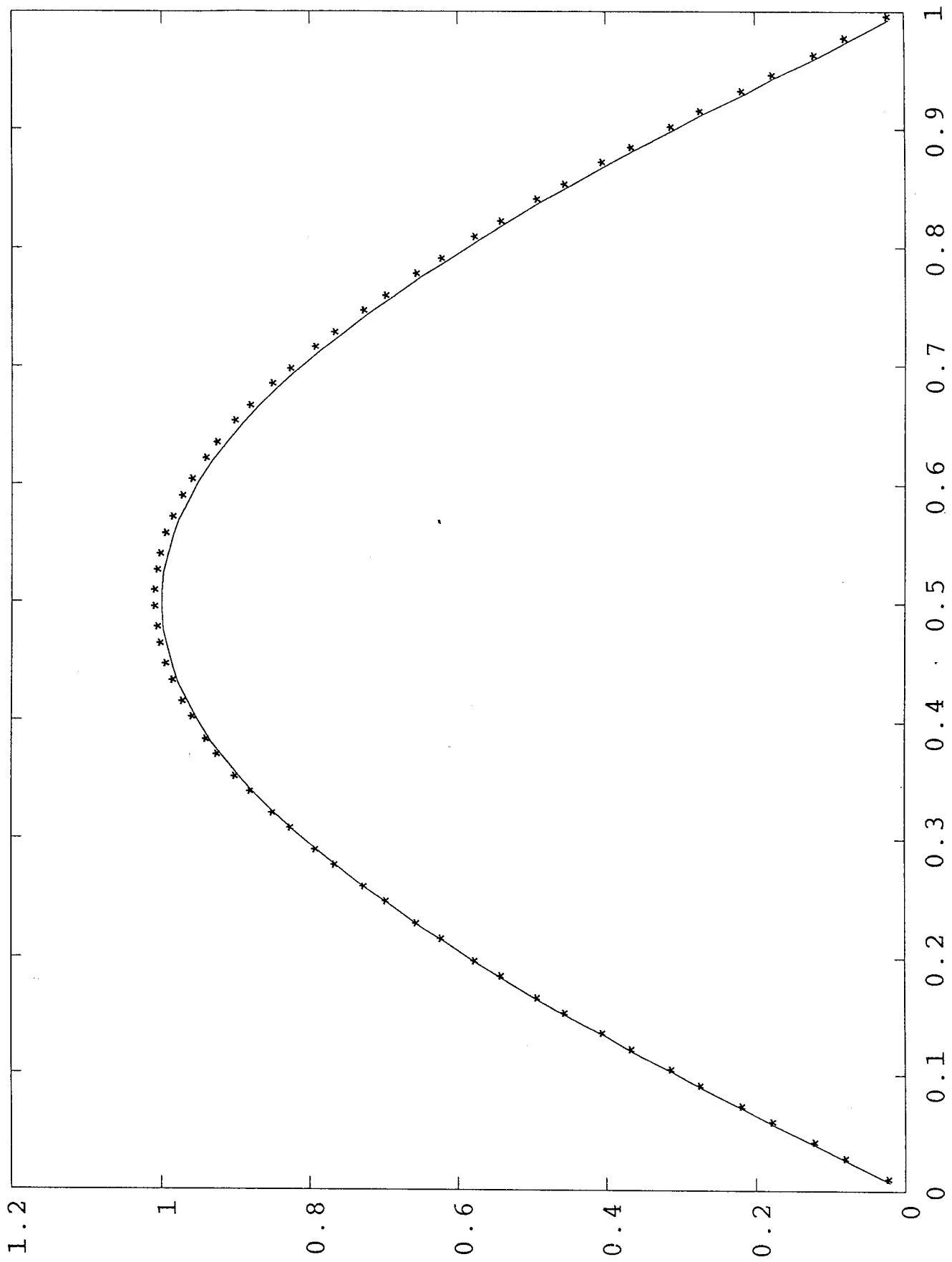


Figure 3.2 (a): Comparison between exact (-) and computed (*) value of $e_0(x_1, .5)$ ($h=1/32$).

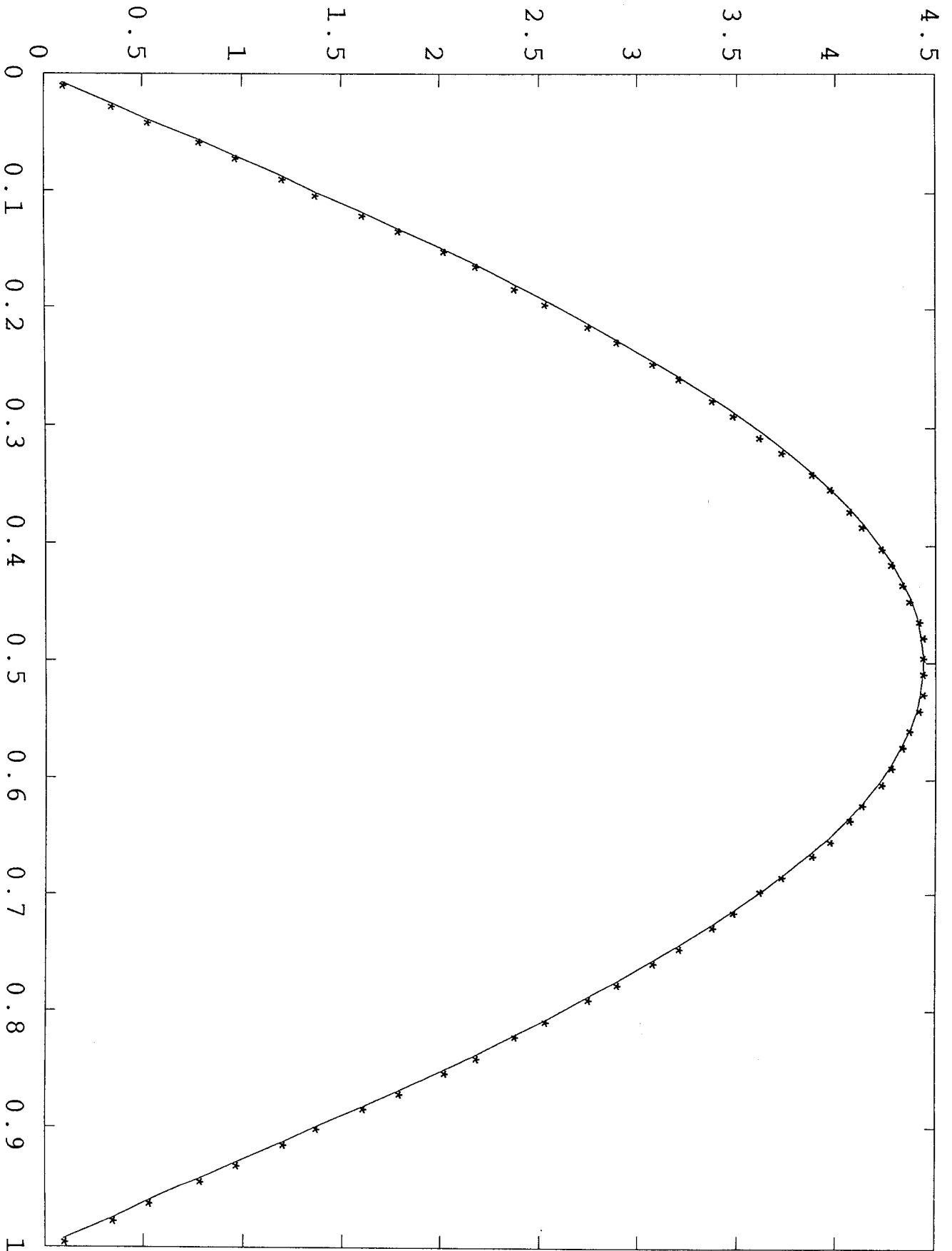


Figure 3.2 (b): Comparison between exact (-) and computed (*) value of $e_1(x_1, .5)$ ($h=1/32$).

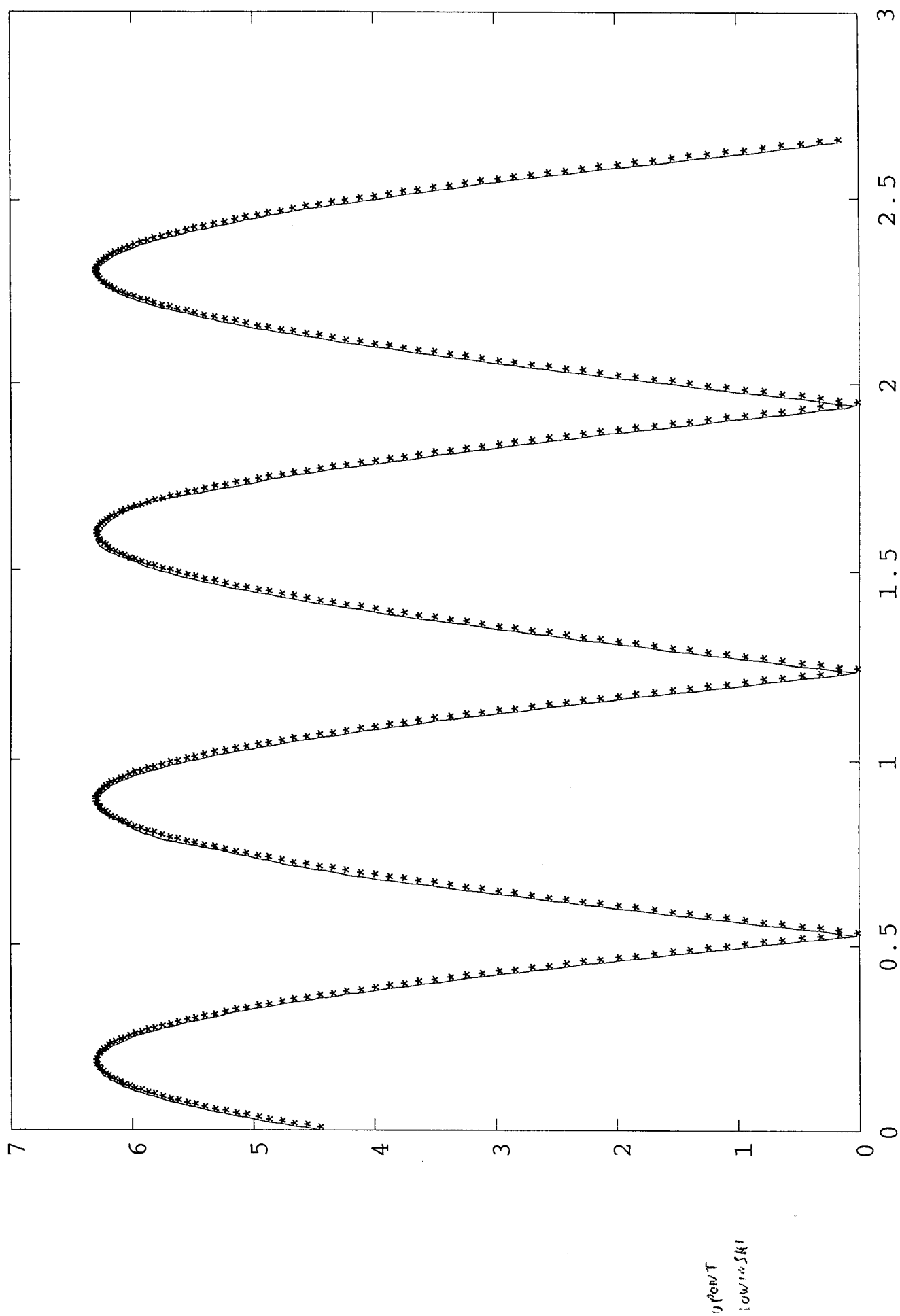
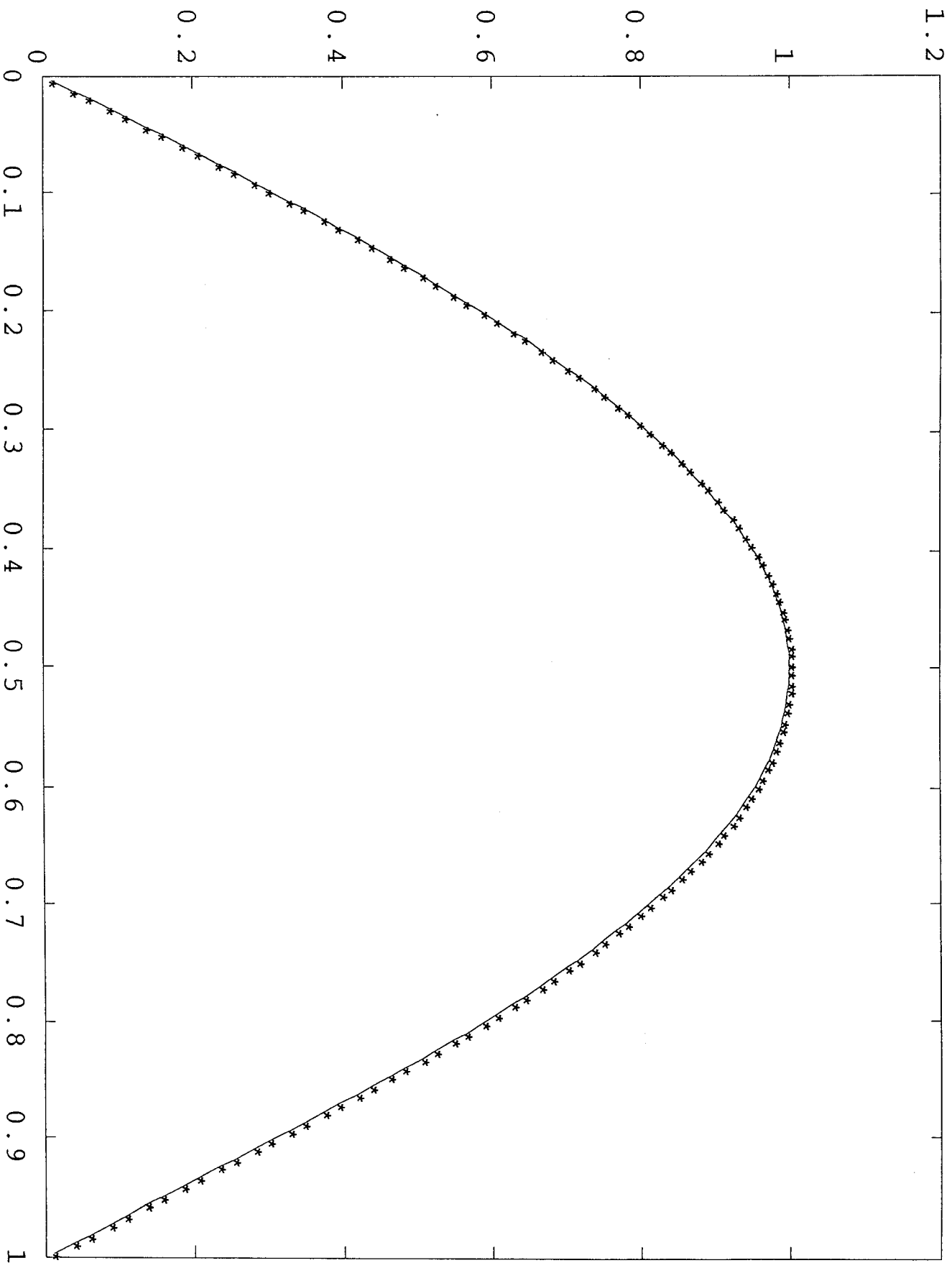


Figure 3.2 (c): Comparison between exact (-) and computed (*) value of $\|g(t)\|_{L^2(\Gamma)}$ ($h=1/32$).



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all

Figure 3.3 (a): Comparison between exact (-) and computed (*) value of $e_0(x_1, .5)$ ($h=1/64$).

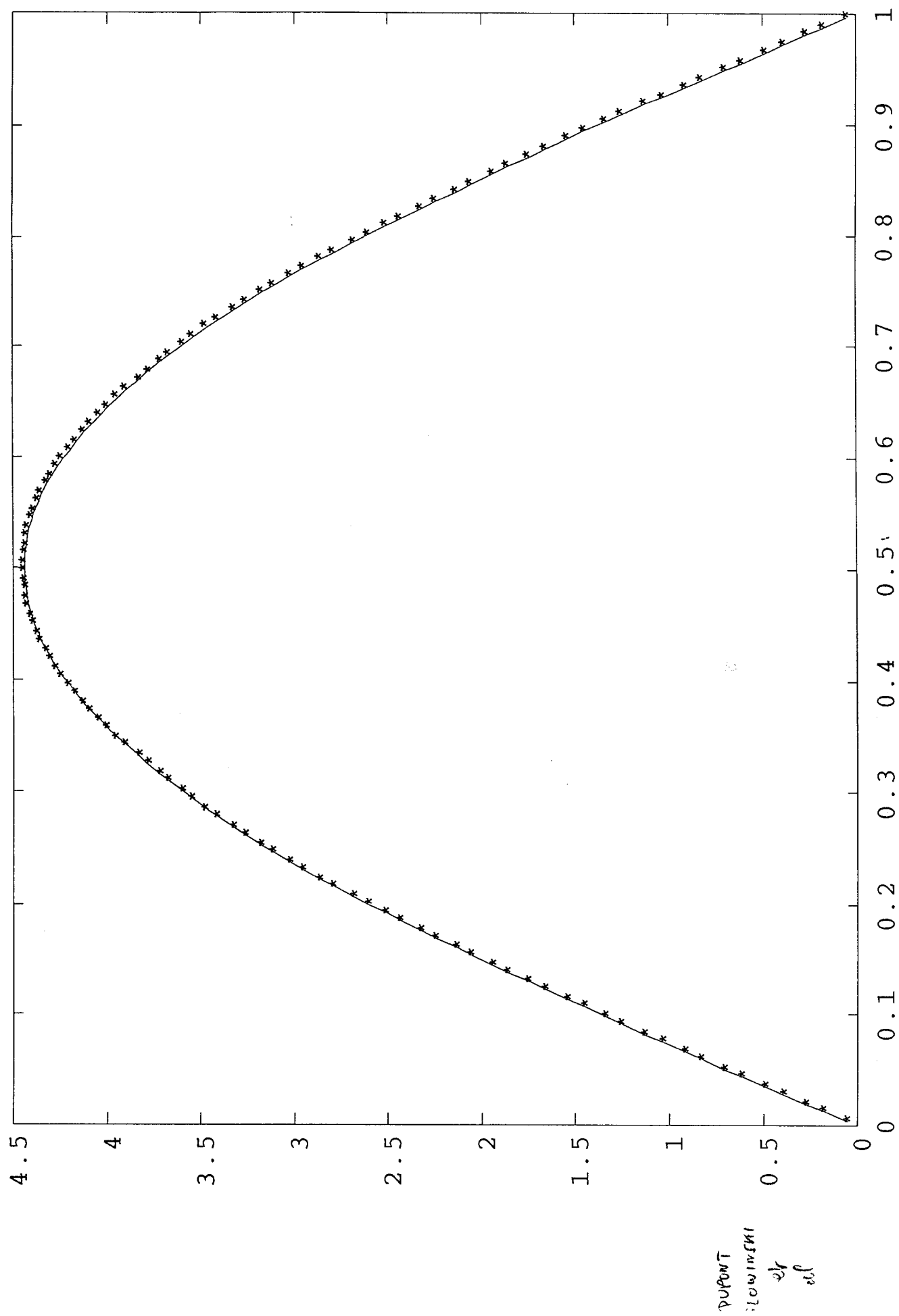


Figure 3.3 (b): Comparison between exact (-) and computed (*) value of $e_1(x_1, .5)$ ($h=1/64$).

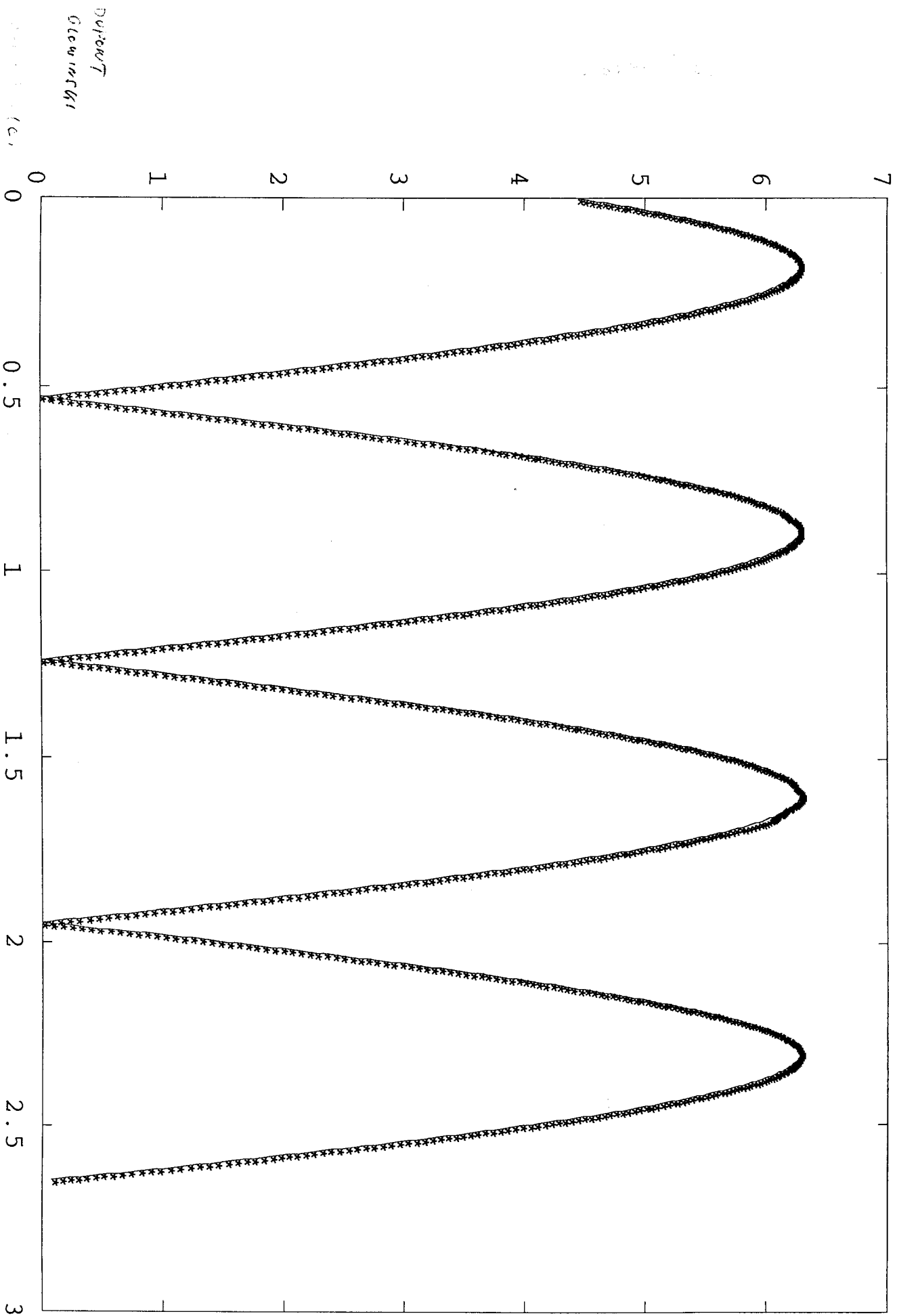


Figure 3.3 (c): Comparison between exact (-) and computed (*) value of $\|g(t)\|_{L^2(\Gamma)}$ ($h=1/64$).